DOCUMENT RESUME

ED 209 080 SE 035 777

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TITLE The Role of Semantic Understanding in Solving Multiplication & Division Word Problems. Final Report.
INSTITUTION Massachusetts Inst. of Tech., Cambridge. Div. for Study and Research in Education.
SPONS AGENCY National Inst. of Education (DHEW), Washington, D.C.
PUB DATE 1 Oct 81
GRANT NIE-G-80-0144
NOTE 49p.; Contains light and broken type.

EDRS PRICE MF01/PC02 Plus Postage.
DESCRIPTORS Basic Skills; Cognitive Processes; Educational Research; *Elementary School Mathematics; Elementary Secondary Education; Learning Theories; *Mathematics Education; Models; *Multiplication; *Problem Solving; *Ratios (Mathematics)
IDENTIFIERS *Mathematics Education Research; *Word Problems

ABSTRACT

The learning of word problems in mathematics is viewed as particularly difficult. The first paper in this report takes into consideration the semantic aspects of problems, by viewing the difficulties that children have solving two-step multiplication problems. An initial study was carried out with 36 children between the ages of nine and fourteen years old; a subsequent experiment involved 93 pupils between the ages of nine and thirteen years. Among the results, the investigations indicated that the concept of ratio was a major source of difficulty in solving problems. Also, being a two-step problem to begin with was a major source of difficulty. It was felt that further study of these two issues could allow for design of teaching strategies to help children surmount these difficulties. The second paper in this document focuses further on the child's development of ratio concepts, with 36 children between the ages of ten and thirteen participating. The results of this investigation served to confirm prior findings that the notion of ratio as a quantity is difficult for children. (MP)

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FINAL REPORT

TO THE

NATIONAL INSTITUTE OF EDUCATION

NIE-G-80-0144

The Role of Semantic Understanding In Solving Multiplication & Division Word Problems

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Division for Study & Research in Education

Massachusetts Institute of Technology

October 1, 1981

U.S. DEPARTMENT OF EDUCATION
NATIONAL INSTITUTE OF EDUCATION
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The work described in the appended two papers was supported by this grant and completed during the course of it.

The first paper, on two step problems, is now being revised in preparation for submission for publication. The second paper, on the development in children of the concept of ratio, is presently being submitted for publication. In addition, the second paper will appear as part of a series of occasional papers published by the Division for Study & Research in Education at MIT.

The Role of Semantic Understanding in Solving Multiplication Word Problems:

Two Step R x E problems

By: Ana Helvia Quintero

The Role of Semantic Understanding in Solving Multiplication Word Problems: Two Step R x E Problems

In mathematics learning word problems are particularly difficult (see the National Assessment of Educational Progress Report 1979). There are numerous studies which try to determine the reasons for these difficulties. In the majority of these studies the categories used for characterizing the word problems are very broad and frequently not informative (e. g. "standardized word problems test", "algebra word problems"). For understanding children's behavior in solving quantitative word problems, there is a need to have a clear description of the types of problems that children are asked to solve. In this description it is very important to take into consideration the semantic aspect of the problem, that is, the meaning of the concepts and relationships involved in the problems. The work done by Carpenter, Hielbert and Moser (1979), Carpenter and Moser (1979), Greeno (1978), Moser (1979) and Nesher (1979), on one-step addition and subtraction word problems, and the work of Quintero (1980), on one-step multiplication work problems, have shown that the concepts and relationships involved in the problem are a strong determiner of problem difficulty for the children.

This study is undertaken with this perspective in mind and is intended to further the understanding of how children solve word problems by studying the difficulties that children have solving two-step multiplication problems of a given structure.

Multiplication models different types of situations. So, in order to understand what type of difficulties children have when solving multiplication word problems there is a need to have a clear description of these different situations.

Distinction among multiplication word problems

In a multiplication word problem each of the factors of the multiplication can represent a different type of quantity. For example in the following word problem:

> "In the store they sell 5 apples per dollar.
> Joan bought 3 dollars worth of apples.
> How many apples did she buy?",

one of the quantities "5 apples per dollar" involves the notion of ratio. On the other hand, we have quantities like "3 dollars" whose measure depends not on the "density" of the referent but on the total "amount". These types of quantities will be called extensive quantities.[1]

According to the type of quantity which the factors represent there are, then, five situations which multiplication can model:

1) a) Extensive x quantifier (E x q)

   like

   5 candies x 3 times as many (candies)

   b) Ratio x quantifiers (R x q)

   like

   5 candies/box x 3 times as many (candies/box)

2) Ratio x Extensive (R x E)

   like

   5 candies/box x 3 boxes

3) Ratio x Ratio (R x R)

   5 candies /box x 3 box/child

4) Extensive x Extensive (E x E)

   5 blouses x 3 skirts (= outfits)

In all of these situations you can have problems of different levels of difficulty by varying the difficulty of the quantities. For example, in the R x E problems you can have a problem like the following:

"In the store they sell 12 candies in each box.
Mary bought 4 boxes of candies.
How many candies does she have?"

where the ratio "12 candies in each box" is a very simple type of ratio; one where the "denominator" is one (this type of problems will be denoted by R x E (d = 1)). On the other hand, you can have a problem like:

"In the store they give you 6 lolly pops for every
2 chocolate bars you buy.
John bought 10 chocolate bars.
How many lolly pops did they give him?"

where the ratio "6 lolly pops for every 2 chocolate bars" is more complex. Here the "denominator" is different from one. (This type of problem will be denoted by R x E ($d \neq 1$)).

Quintero (1980) studied the type of difficulty that children have solving R x E (d = 1) problem. She found that children's major difficulty with this type of problems was in understanding the relationship among the elements that form the ratio. She also observed that the majority of the children who sucessfully worked R x E (d = 1) problems did not successfully work problems of the R x E ($d \neq 1$). This study will explore the types of difficulty that children have solving R x E ($d \neq 1$) problems in order to ascertain in what respect they coincide with the difficulties in R x E (d = 1) problems and in what respect they differ.

Solving R x E problems (either with one or another number in the "denominator") involves various components: understanding the relationship between the elements that form the ratio (R_1); understanding the relationship between the ratio and the extensive quantity (R_2); and mapping of the latter relationship to a mathematical operation with the same structure (f).

This study will then attempt to analyze which of the components mentioned above (R_1, R_2, or F) is a source of difficulty when children are solving R x E ($d \neq 1$) problems pharased in the following way:

There are a x's for every b y's

If there are c y's.
How many x's are there?

Three R x E ($d \neq 1$) problems were presented to 9 to 14 years old children. They were asked to repeat the problems and to pick the correct representations of the problems among a set of drawings. Besides the three R x E ($d \neq 1$) problems children were asked to solve an R x E ($d = 1$) problem; a two step problem involving only ratios with one in the "denominator", and two fillers, a simple addition and subtraction problem.

The fillers were used to identify children who had adopted some fixed solution strategy regardless of the actual problem. The R x E ($d = 1$) problem was used to test the observation made in our previous study; that R x E ($d = 1$) are easier than R x E ($d \neq 1$) problems. Comparing the performance on the various tasks of children who solved the R x E ($d = 1$) problem correctly with those that work it incorrectly can be an instrument to determine the different levels of understanding of the notion of a ratio.

The repetitions and drawings picked by the children were analyzed for evidence of systematic confusions. The results shed light on which of the components of solving the R x E ($d \neq 1$) problems can account for the difficulty that children experience with them.

The presence of errors in representation of the ratio (R_1) and/or of the relationship between the ratio and extensive quantity (R_2) in the drawing that children picked or in the repetition indicates that the representation of these relationships is a possible source of difficulty for the R x E problems.

On the other hand if the drawings picked by the children and the repetition show a correct representation of the relationships, but children still solve the problems incorrectly, then the difficulty must lie in mapping the structure of the problem to a mathematical operation with the same structure (F). This may be affected by these problems being two-step problems. The two step problem with

ratios with one in the denominator is designed to help determine how being a two step affects the difficulty of the R x E ($d \neq 1$) problems.

There are two ways of working the two steps in the R x E ($d \neq 1$) problems. For example, in the following problem:

> "In the store they give 6 lolly pops to every 2 chocolate bars you buy.
> John bought 10 chocolates bars.
> How many lolly pops did they give him?"

one can either simplify first the ratio; 6/2 = 3 lolly pops for chocolate bar and then multiply that quantity by the number of chocolate bars; or on the other hand one can first look for how many groups of 2 chocolate bars ther are in 10 chocolate bars (10/2 = 5) and then multiply that number by the number of lolly pops for 2 chocolate bars. In the first case the problem is reduced to a one-step R x E problem, while in the second case the problem is reduced to a one-step E x q problem.

The nature of the numbers used in the problems could affect the method used by the children. For example, if in the above example we use numbers such as "6 lolly pops for every 2 chocolate bars, and bought 7 chocolate bars"; that is, only a/b is an integer, it is expected that children will reduce this problem to a one-step R x E problem. On the other hand, if the following numbers are used: "7 lolly pops for every 2 chocolate bars and bought 6 chocolate bars; that is, only c/b is an integer, it is expected that children will reduce the problem to a one-step E x q problem. In other words, it is expected that children will first do the operation where an integer is obtained. Now, if children use one method, regardless of the numbers used in the problem, they must be using one representation for the problem, which is determining the method to solve it.

Experiment I

Method

Subjects

The study was carried out with 36 children between 9 and 14 years of age.

(9 years N = 2, 10 years N = 11, 11 years N = 11, 12 years N = 9, 14 years N = 2) The children were students of the Julio Sellés Solá (N = 12) and Rafael López Sicardó (N = 12) Public Elementary Schools, and of the Barbosa (N = 6) and Sotero Figueroa (N = 6) Public Intermediate Schools. There were 12 children from each of the 5th, 6th and 7th grades. There were 17 boys and 19 girls.

Procedure and materials

The experimenter worked two tasks with one child at a time. The session lasted around half hour and was audio-taped. In the first part the children solved and repeated a set of word problems. In the second part the children were asked to choose among a set of drawings the ones that represented the situation described in the R x E (d = 1) and R x E (d ≠ 1) problems. The problems for the drawing task were presented in the same order that they were solved.

Task I: Problems

In this task the children solved a total of seven problems: three R x E (d ≠ 1) problems, an R x E (d = 1) problem, a two-step problem which only involved ratios with one as "denominator" and two fillers: a simple addition and a subtraction problem.

The filler and the R x E (d = 1) problem were the same for all the children. The order of presentation of these problems was also the same for all the children. They were the second (addition problem), fourth (subtraction problem) and sixth (R X E (d = 1) problem) problems.

The R x E (d ≠ 1) problems varied both in context and in the numbers used. There were three contexts: lollypops/ chocolate bars, handkerchief/$ and children/teacher; and three sets of numbers: only a/b was an integer, only c/b was an integer; a/b and c/b were both an integer (see Apendix A). An effort was made to have the contexts of similar familiarity for the children. Each context was presented with each set of numbers. The order of presentation of the problem

was then randomized for each subject.

The two-step problem was always the third problem. It's context correspond to the problem where only a/b was an integer.

The problems were presented one by one to the child. If the method by which he had solved the problem was not clear to the experimenter, she asked him to explain how he had worked it. Once each problem was solved the experimenter read the problem and asked the child to repeat it.

Task II: Model Recognition

The children were asked to choose among a set of drawings the one that represented the situation described in the R x E (d = 1) and R x E (d ≠ 1) problems which they had just solved. Some of the drawings correctly represented the problems while others represented it incorrectly. The drawings were made by analyzing the possible representations, correct or incorrect, that the child could make of the problem. This analysis was complemented by having a group of children freely draw representations of the problems during a pilot study.

There were nine drawings with common structure for all the R x E (d ≠ 1) problems. There were three drawings showing the correct representation of the ratio; one of them was also a correct representation of the problem, while the other two didn't show correct relationships between the ratio and the extensive quantity. Two drawings reduced the R_1 relationship from x/y to x/1; one of them represented just that R_1 relationship while the other also expressed the relationship between the ratio and extensive quantity. The other four drawings represented one or both of the elements that enter to form the relationships of the problem but not the relationships.

In the problems where a/b was an integer there was an extra correct drawing representing the method of simplifying the ratio first, and in the problems where c/b was an integer there was an extra correct drawing representing the R x E as E x q (see figure 1.)

Insert figure 1 around here

In the R x E (d = 1) problem there were seven drawings. Four of them have a representation of a relationship between the ratio and extensive quantities. One of them was a correct representation, while the rest incorrectly represented the ratio; the relationship between the ratio and the extensive quantity, or both. The other three represented either one or both of the elements that enter to form the ratio (see figure 2).

Insert figure 2 around here

The experimenter displayed the drawings in front of the child together with the word problem that they represent and told the child:

Among these drawings there are some which correctly represent this problem and some which don't. Please pick the correct ones and tell me why are they correct.

Results

Fillers

The fillers were worked correctly[2] all the time.

Performance levels

R x E (d = 1) problem

Twenty nine out of thirty six children (80.5%) worked the R x E (d = 1) problem correctly. Of the 29 children that worked the problem correctly, 21 (72.4% of the ones that work it correctly, 58.3% of the total) worked the problem multiplying 12 x 4. The rest added 12 four times.

R x E ($d \neq 1$) problem

The performance of the children on the R x E ($d \neq 1$) problems was significantly (Chi Square Test $p \leq .001$) lower than their performance on the R x E (d = 1) problems.

There was no significant differences on the performance of the children on the problems of different context nor on the problems of different numbers (see

table 1).

Insert Table 1 about here

The problems that were solved correctly were always solved by looking how many b were in c (this was sometimes done by direct division, other by times by adding b until it gets to c) and then multiplying that quantity by a (or adding a that number of times). This led to the result that the problem were only a/b was an integer was sometimes solved by approximation. For example there was a child (11 years, F, 6th grade) that when working the problem:

> In a store they give 6 lolly pops for every 2 chocolate bars.
> John bought 7 chocolate bars.
> How many lolly pops were given to John?

said:

> 2 chocolate bars——————— 6 lolly pops
> 4 chocolate bars——————— 12 lolly pops
> 6 chocolate bars——————— 18 lolly pops
>
> So it will be around 20 lolly pops.

Half of the times that the problems where only a/b was an integer was worked correctly was done in a way involving some type of approximation.

Two step problem

Only two children correctly worked the two step problem involving only ratios with one as "denominator". The R x E ($d \neq 1$) problems involving the ratio, handkerchief/$ and children/teacher, were solved significantly better (Chi Square Test $\leq .05$) than the two-step problem There was no significant differences between the lolly pops/ chocolate bars and the two-step problems.

Repetition

Fillers

The fillers were repeated correctly almost all the time. Only once a child couldn't remember the subtraction problem.

R x E (d = 1) problem

Relation between repetition and performance

A close relationship between being able to solve the problem correctly and being able to repeat the problem correctly is found. Thirty three out of 35 times[3] (94.2%) the problem and the repetition were both correct or both incorrect.

Type of error in the repetition

The error in the repetitions were analyzed. Four out of the five errors were a change from a ratio to an extensive quantity, for example, "candies per bag" to "candies". The other error was that the extensive quantity was changed from boxes to candies.

R x E ($d \neq 1$) problems

Relation between repetition and performance

The relationship between being able to solve the problem correctly and being able to repeat the problem correctly is not as strong in this case as in the R x E ($d = 1$) problem. Sixty nine times out of 103 (66%) the problem and the repetition were both correct or both incorrect. A considerable number of times (32%), children worked the problem incorrectly but could repeat it correctly.

Type of error in the repetition

The errors in the repetition were analyzed. Six responses in which either only the situation is described or it is not clear if the repetition was correct or incorrect, are not included.

As in a previosu study (Quintero, 1980) the errors can be divided in two broad categories: omission errors, in which the child leaves out one or both premises of the problem; and distortion errors, where some part of the problem is changed. Distortion was more common than omission errors (see Table 2). More important, out of the distortion errors, 23 (74%) consisted in a change in the ratio. There were two ways in which the ratio was changed. In one case it

was changed from a ratio to an extensive; for example, from "5 lolly pops for every 4 chocolate bars to "5 lolly pops". In the other case, the ratio was changed to one with one in the "denominator"; for example, from "5 lolly pops for every 4 chocolate bars" to "5 lolly pops for each chocolate bar".

Insert Table 2 about here

Two- step problem

Relation between repetition and performance

A close relationship is found again between being able to solve the problem correctly and being able to repeat the problem correctly. Twenty nine out of 32 times (90.6%) the problem and the repetition were both correct or both incorrect.

Type of errors in the repetition

Contrary to the R x E (d = 1) and R x E (d ≠ 1) problems the major type of error in the two-step problems was the omission error. Twenty one out of 25 errors (84%) were omission errors. The children forgot either one, two or all the three premises. The premise which is most often forgotten is the second one.

Model Recognition

R x E (d = 1) problem

Twenty four of the 29 children (82.7%) who worked the R x E (d = 1) problem correctly picked the drawing representing correctly the problem. Besides picking the correct drawing some children (N = 7, 29%) also picked a drawing which only represented part of the problem correctly. The R x E (d = 1) problem talked about 12 candies in each box and 4 boxes of candies. Some of these children, for example, picked a drawing showing four boxes, since the problem stated that "María had bought 4 boxes of candies".

Of the seven children who worked the problem incorrectly five (71.4%) picked only incorrect representations, one picked both correct and an incorrect repre-

sentations, and only one (14%) picked only the correct representation. Of the five children who picked only an incorrect representation, four picked the card showing the elements that enter into the problem but not the relationship and one picked all the cards with 12 boxes. The child who picked the correct and incorrect representation picked all the cards with the correct ratio.

R x E ($d \neq 1$) problems

Eighty seven percent of the time, solving the problem and picking drawings representing the problem were both correct or both incorrect. When picking the drawings the children could pick one or several drawings. All the children that picked the drawings correctly, chose the drawing representing both relationships as they were given in the problem (see figure 1c). Only 17 percent of the time did the children in this group pick also the drawing representing the method of simplifying the ratio first (see figure 1k); seven percent of the time they also picked the drawing representing R x E as E x q (see figure 1e).

Some children in this group, besides picking the correct representation, picked also an "incorrect" representation. Yet, when they gave the reasons for picking the "incorrect" drawing one could see that they interpreted the drawing in a way that made it a correct representation. This fact points to the importance of having children explain their answers.

The errors that children made while picking the drawings can be divided into several categories:

(1) Children who chose cards where one or both of the elements mentioned in the problem were present but not the relationships (39% of the time) (see figure 1 f, g, h, i).

Some children in these groups picked just one card but others, for example, picked all the cards in which there were a total of 15 lolly pops.

(2) Children who changed the ratio to an intensive quantity with one as a denominator (10% of the times) (see figure

1 d, e).

(3) Children who picked all or some of the cards in which the ratio was correct, but not necessarily the relationship between the ratio and extensive quantity (31% of the times) (see figure 1 a, b)

Among this group 32 percent of the times the children picked only the card that showed the ratio alone (see figure 1a); 64 percent of the times they picked all the cards which showed the correct ratio, among them the correct card; and 4 percent of the times they picked just the card showing the correct ratio but not the extensive quantity (see figure 1b).

(4) Children who picked one card representing the first premise and another one representing the second one (12.5% of the times).

(5) Children who picked several cards and gave different reasons for each one. Among this group there may have been children who used the criteria of category (4) but didn't express it clearly(10% of time).

Discussion

R x E (d = 1) problem

In this study there was a small group of children who solved the R x E (d = 1) problem incorrectly. The performance of these children on the repetition and the model recognition task showed that the major source of error in solving the R x E (d = 1) problem was in understanding the relationship between the elements that form the ratio (R_1). In the repetition four out of the five errors were a change from a ratio to an extensive quantity. Also, in the model recognition the major error consisted in picking the cards where neither R_1 or R_2 were represented. Only once did a child picked a card were R_1 was correct but not R_2. These results confirm the results of Quintero (1980) that the major difficulty

in solving R x E (d = 1) problems is in understanding R_1.

R x E (d ≠ 1) problems

The great majority of the children had difficulty with the R x E (d ≠ 1) problems. The difficulties that the children had did not all stem from the same source.

There was a group of children that had difficulty in understanding the ratio (R_1). In the repetitions 74 percent of the distortion errors consisted of repeating the ratio incorrectly. In the model recognition task 49 percent of the time that children had difficulty they did not represent the ratio correctly. The nature of this difficulty was not the same for all these children. Some children changed the ratio to an extensive one in both the repetition and drawing. This group of children coincides with the children who worked the R x E (d = 1) problem incorrectly. Another group of children changed the ratio of the problem to a ratio with one in the "denominator". All the children in this group, except one, had worked the R x E (d ≠ 1) problem correctly.

Another source of difficulty in solving the R x E (d ≠ 1) problems, was understanding the relationship between the ratio and the extensive quantity (R_2). In the repetition a group of children repeated the ratio correctly but changed the extensive quantity (26 percent of the time). In the drawings, 31 percent of the times the children picked cards showing the correct ratio but not the correct relationship between ratio and the extensive quantity. In this respect the difficulties that children had with R x E (d = 1) problems and with the R x E (d ≠ 1) problems differed. In the former case a small percentage of the children either repeated (16.7%) or picked a card (4.2%) with a correct ratio but with the incorrect relationship between ratio and the extensive quantity (see Quintero 1980), while in the latter case there was a considerable number of children who seemed to understand R_1 but not R_2. All the children who showed no difficulty with R_1

in the R x E ($d \neq 1$) problems, had worked the R x E ($d = 1$) problem correctly.

There seems to be, then various stages in the understanding of the concepts and relationships involved in the R x E problems:

(1) First, children understand simple ratios with one in the "denominator". In this stage children can solve R x E ($d = 1$) problems but not R x E ($d \neq 1$) problems.

(2) Children understand ratios with "denominator" different from one, but not the relationship between this ratio and the extensive quantity.

(3) Children understand the relationship between ratio with "denominator" different than one and an extensive quantity (R_2).

It could be argued that the difficulty with R_2 in the R x E ($d \neq 1$) problems is not due to a conceptual misunderstanding, but to a difficulty in organizing the information of the problem, given that these problems are two-step problems. As a matter of fact, the two-step problem involving only intensive quantity with one in the denominator was more difficult than the R x E ($d \neq 1$) problems.

Yet, the two-step problem involving only ratios with one in the denominator involved more premises than the R x E ($d \neq 1$) problems, and so it involved more information to be organized. Indeed, the type of error in the repetition of the two-step problems involving only ratios with one in the denominator were typically omission errors (84%), while in the R x E ($d \neq 1$) problems the most common error was the distortion error (68.8%). It must be proved with two-step problems having the same number of premisses as the R x E ($d \neq 1$) problems and involving simpler concepts whether being a two-step problem per se is the cause of the difficulty with R_2 in the R x E ($d \neq 1$) problems.

Experiment II

Method

Subjects

The study was carried out with 93 children between 9 and 13 years of age

(9 years N = 1, 10 years N = 29, 11 years N = 27, 12 years N = 31, 13 years N = 4).[4] The children were students of the Julio Sellés Solá (N = 67) and Sotero Figueroa (N = 26). There were children from 5th (N = 32), 6th (N = 35) and 7th (N = 26) grade. There were 31 boys and 42 girls (20 children didn't write down their sex).

Procedures and materials

The experimenter worked with the children in their classroom; one group of fifth grade, a group of sixth grade, and a group of seventh grade. She gave each child a set of four two-steps problems to solve. She asked them to write down the operations used to solve the problems.

Among the four problems there were two R x E ($d \neq 1$) problems and two two-step problems which didn't involve ratios (denoted from now on as two-step problem).

The R x E ($d \neq 1$) problems varied in context and numbers used. There were two contexts: lolly pops/chocolate bars and children/teacher. In the children/teacher problem a/b and c/b were both integers, while in the lolly pops/chocolate bars problem only c/b was an integer (see Appendix B).

The two-step problems also varied in context and number. Furthermore the operations used to solve the problems were different. In one case one has to multiplying a, b and then add c (this problem will be denoted, Two-step (x,+)). In the other problem one has to multiply a and c then subtract b (Two-step (x,-), see appendix B).

The problems were presented in two ways:

	Form I	Form II
1st problem	R x E ($d \neq 1$); a/b and c/d=n	Two-step (x, +)
2 nd problem	Two-step (x, +)	R x E ($d \neq 1$); a/b and c/d=n

	Form I	Form II
3rd problem	R x E (d ≠ 1), only c/d=n	Two-step (x, -)
4th problem	Two-step (x,-)	R x E (d ≠ 1), only c/d=n

Results

The children had more difficulty solving the R x E (d ≠ 1) problems than the two-step problems. The difference was significant (Chi Square Test $p \leq .001$). There was no significant difference between the performance on the two R x E (d ≠ 1) problems, nor between the performance on the two-step problems (see Table 3).

Insert Table 3 about here

The method followed by the students when they solved correctly the R x E (d ≠ 1) problems was similar to the method followed in Experiment I, they reduced the problem to a one-step E x q problem.

Dicussion

In this experiment children were asked to solve two-step problems, involving simple concepts and R x E (d ≠ 1) problems. Children performed significantly better in the two-step problems than in the R x E (d ≠ 1) problems. This implied that children's difficulty with the R x E (d ≠ 1) problems is not confined to their being a two-step type of problem.

The fact that a considerable number of children (37.6% in the (x,+) problem and 40.6% in the (x, -) problem) had difficulty with the two-step problems which involve very simple concepts implies that being a two-step problem is also a source of difficulty for children when solving word problems.

General conclussions

This study shows that a major source of difficulty in solving R x E ($d \neq 1$) problems is understanding the concept fo a ratio. There is also evidence that there are levels of understanding of this concept. There is need, however, to study further the different levels of understanding of the ratio concept and how they affect the way the children solve R x E problems.

The fact that the concept of ratio, which is not common to all types of multiplication word problems; was the main source of difficulty in solving R x E problems shows the importance of having precise categories to characterize the word problems that children are asked to solve. If this work had studied children's difficulties solving multiplication problems without looking at the different situations which multiplication models it may have overlooked children's difficulty understanding ratios. Clearly the difficulty children will have with other type of problems, such as E x E problems, is of another nature.

Experiment II shows that being a two-step problem is also a source of difficulty when solving word problems. Yet, it is not clear from this study the specific role that being a two-step problem plays in children's difficulty with the R x E ($d \neq 1$) problems. It is, then, important to explore the nature of the difficulties introduced by two-step problems.

Studying these issues, the role of being a two-step and the levels of understanding of a ratio, will permit us, not only to understand better children's difficulty with word problems, but will allow us to design strategies for helping children surmount these difficulties.

Notes

1) For a more extensive discussion of these concepts see Schwartz (1976) or Quintero (1980).

2) In this and in all our analysis, "correct" means that the problem is posed correctly, there may be a computational error.

3) A child is not included because it was not clear if his repetition was correct or incorrect.

4) One child didn't write down his age.

5) Two children's results are not included since it was not clear if their error was computational or conceptual.

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APPENDIX A

Problems used in the Study
(The original problems were in Spanish)

Fillers:

1) In a classroom there are 16 boys and 18 girls. How many children are in the classroom?

2) Charlie had 12 baseball cards.
He gave 5 to Bobby.
How many does he has left?

I x E (d = 1) problem

In the store they sell 12 candies in each box. María bought 4 boxes of candies. How many candies will María have?

Two-step problem (corresponding to the candy content)

In a store they sell 24 candies in each box. For a birthday party they are going to make 4 surprise bags from each box of candy bought in the store. If they give 3 surprise bags to each child in the birthday party, how many candies will each child have?

I x E (d ≠ 1) problems

1) In a store they gave a lollypops for every b chocolate bars that one buys. John bought c chocolate bars. How many lollypops were given to John?

2) In a store they sell a handkerchief for b dollars. On Monday they sold a total of c dollars in handkerchief. How many handkerchiefs were sold on Monday?

3) In an art class they can admit a students for every b teachers. They have c art class teachers. How many students can they admit.

(a, b, c) take value (6, 2, 7), (5, 4, 12), (15, 3, 12)

APPENDIX B

Problems used in Experiment II
(the original problems were in Spanish)

Two step problems

(x, +) Richard is 7 year old.
Peter is 5 times as old as Richard.
John is 4 years older than Peter.
How old is John?

(x, -) Charlie has 5 marbles.
Robert needs 3 marbles to have three times the number of marbles that Charlie has.
How many marbles does Robert have?

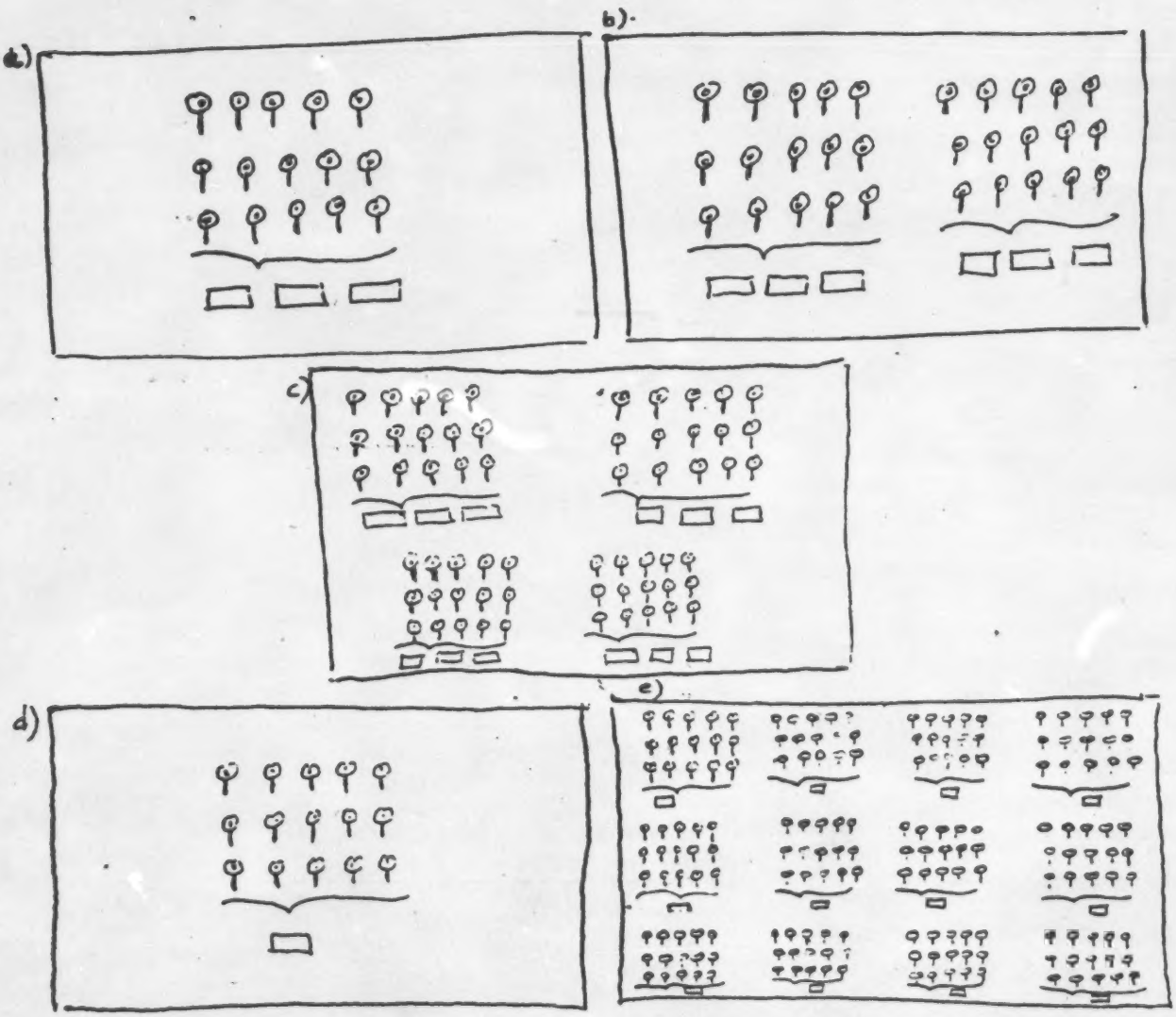
I x E (d ≠ 1) problems

In an art class they can admit 15 students for every 3 teachers.
They have 12 teachers.
How many students can they admit?

In a store they gave 5 lolly pops for every 4 chocolates bars that one buys.
John bought 12 chocolate bars.
How many lolly pops were given to John?

Figure 1: Drawings for the Model Recognition Task

R x E ($d \neq 1$) problems



Problem: In a store they gave 15 lolly pops for every 3 chocolate bars that one buys.
John bought 12 chocolate bars.
How many lolly pops were given to John?

f)

g)

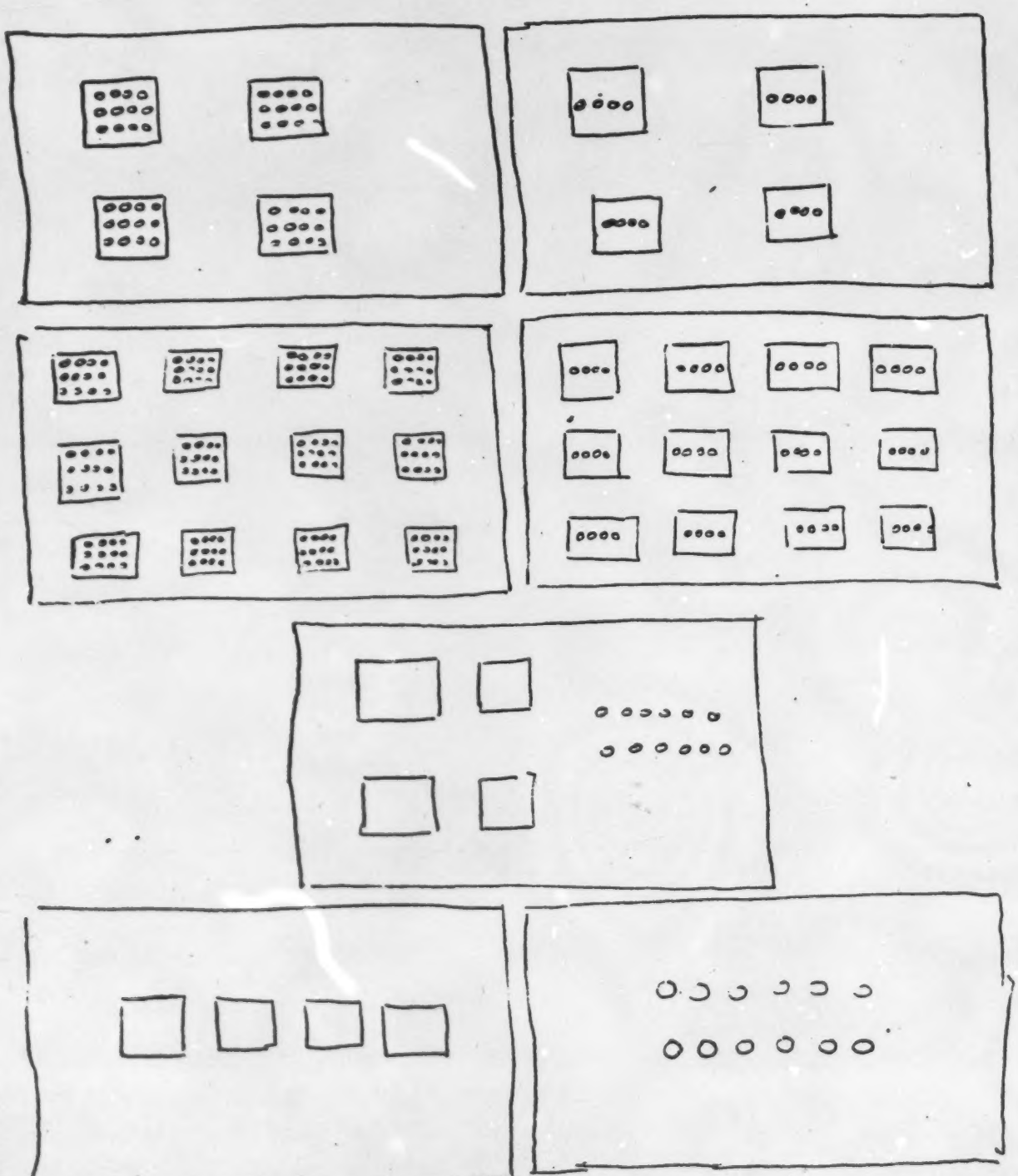
h)

i)

k)

l)

Figure 2: Drawings for the Model Recognition Task
R x E (d = 1) problems



Problem: In a store they sell 12 candies in each box.
María bought 4 boxes of candies.
How many candies will María have?

Table 1

Performance on R x E ($d \neq 1$) problems

Problems by context			Problems by numbers		
	Correct	Incorrect		Correct	Incorrect
lolly pops/choc. bars	5	31	a/b an integer	6	30
handkerchiefs/$	8	28	b/c an integer	6	30
children/teacher	8	28	Both integer	9	27

Table 2

Type of error when repeating the R x E (d ≠ 1) problems

	Type of error	Number of times
Omission errors	Leave out premise involving intensive quantity	3
	Leave out premise involving the extensive quantity	5
	Leave out both premises	6
	Total	14
Distortion errors	Changes Intensive to extensive	8
	Changes x/y to x/1	15
	Intensive correct changes extensive	8
	Total	31

Table 3

Performance on Experiment II

	C	I
R x E ($d \neq 1$); a/b, c/b = n	25	68
R x E ($d \neq 1$); only c/b = n	29	64
Two-step (x, +)	58	35
Two-step[5] (x, -)	54	37

THE DEVELOPMENT OF THE CONCEPT OF RATIO IN CHILDREN

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The concept of a ratio is difficult for children to understand. Numerous studies (Karplus & Peterson, 1970; Wollman and Karplus, 1974; Karplus etal. 1977; Lowell & Butterworoth, 1966; Lunzer and Pumfrey, 1966; Pulos etal, 1981) have revealed that a substantial fraction of secondary school students did not use proportion successfully to solve simple ratio tasks. Inhelder and Piaget (1958) affirm that "in all realms (space, speed, chance, etc.) the notion of proportion does not appear until formal substage III-A".

These previous studies notwithstanding, it is our hypothesis that some elementary school students have reasonable notions of proportion. For example, in preparing a party they know that if we double the number of children we should double the number of snacks.

Their notion of proportionality, however, is not completely developed. We believe that concepts are not grasped in their entirety and thus do not entail fixed states, but include both progressive shifts in understanding and various processes necessary to accomplish those shifts.

The objective of the research that we reporting on here was to study the development in children of the concept of ratio, and in particular the transition from qualitative cognitive models of ratio to quantitative ones.

The Problem

A ratio is a relationship between two quantities which can be used to represent either a homogeneous or heterogeneous entity. In the case of heterogeneous entities the ratio usually represents an average. For example, if we mix sugar and water and stir the mixture well, the ratio of sugar to water (sweetness) will be the same for any portion of the mixture. In this case the ratio represents a homogeneous sample. On the other hand, if we say that in our school system there are 30 students per teacher we don't expect to find this to be true of each of the classrooms. To drive the point home, consider a country in which the average number of children per family is 2.5! Clearly,the ratio 2.5 children/family can have meaning only as an average.

RESEARCH PLAN

A first step toward understanding children's notions of ratio is to find out how they understand the relationship among the elements that enter into the formation of a ratio. This relationship can vary in complexity. We begin by considering children's understanding of relationships that represent homogeneous samples since we consider them simpler than heterogeneous samples that must be interpreted as averages.

One way to probe children's understanding of the relationship among the elements that form a ratio is to see if children can tell in what way a change in magnitude of one or both of the elements that enter into the relationship affects the magnitude of the ratio. Simply stated, a ratio is a magnitude. If this sort of magnitude is to be understood, it could be argued that the ability to compare and order such magnitudes is primary. It is to this issue that we turn our attention.

The magnitude of a ratio can be changed in various ways. These ways of changing the ratio can be classified in two general categories: simple changes and coordinated changes. The simple changes involve a change in only one of the elements entering in the relationship. Symbolically:

$$\frac{N}{D} \quad \text{changes to} \quad \frac{N+d}{D}, \ \frac{N-d}{D}, \ \frac{N}{D+d}, \ \frac{N}{D-d}.$$

For example, in a mixture of sugar and water a <u>simple</u> change of the ratio of sugar to water (sugar/water) could be made by adding to (or subtracting from-not always possible to do) an amount of sugar (or water, but not both) to the mixture.

It is of central importance to note that in the case of <u>simple</u> changes in ratios, it is possible to decide whether the magnitude of the ratio has increased or decreased using <u>qualitative</u> arguments alone. For example, if we make a solution of sugar and water by dissolving sugar in water in the ratio of 3 tablespoons of sugar per cup of water, and if we change this ratio by increasing the amount of sugar per cup of water, we know that this will increase the ratio of sugar/water even if we don't know by how much.

The situation with coordinated changes is more delicate. Coordinated changes consist of simultaneous changes of both elements of the relationship. In the case of coordinated change it is necessary to make a further distinction: coordinated changes in which the order properties of the ratios can be decided qualitatively, such as:

$$\frac{N}{D} \quad \text{changes to} \quad \frac{N+d}{D-d'} \quad \text{or} \quad \frac{N-d}{D+d'}$$

and coordinated changes where the order properties of the ratios can only be decided quantitatively, such as:

$$\frac{N}{D} \quad \text{changes to} \quad \frac{N+d}{D+d'} \quad \text{or} \quad \frac{N-d}{D-d'}$$

Asking children to order ratios can give insights about how they interpret the way in which changes in the magnitude of one, or both, elements of the relationship affect the magnitude of the ratio. This, in turn can help us to understand children's conceptions of the relationships among the elements that enter into the formation of a ratio.

Children were asked to order ratios where a decision can be reached using qualitative attributes (we will denote these as qualitative comparison) like the

ones in the following table:

TABLE I

Comparison that can be based on qualitative judgement

Compare:	
3/1 with 2/1	6/2 with 5/2
3/1 with 3/2	6/2 with 6/3
3/1 with 2/2	

They were also asked to order ratios in which a quantitative judgement was necessary. For example:

TABLE II

Comparison that requires quantitative judgement

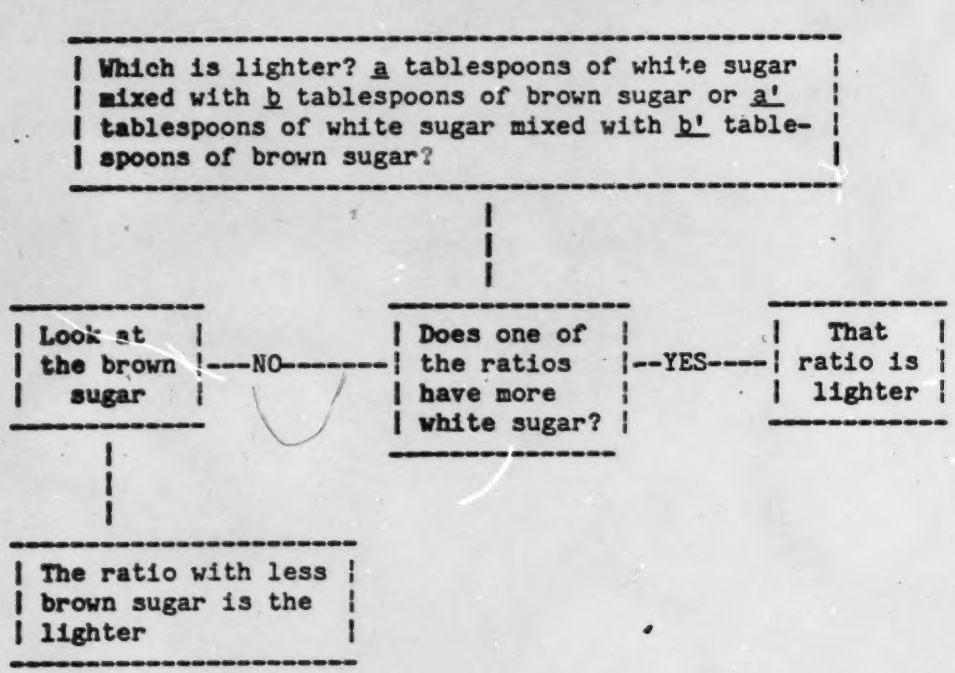
Compare:	
3/1 with 5/3	6/2 with 8/4
3/1 with 6/2	6/2 with 9/3
5/3 with 7/5	
5/3 with 3/2	

It is our hypothesis that the earliest notions of ratio that children develop are based on the qualitative attributes of the relationship. The notion of a ratio as a quantity takes longer to develop. It is to be expected then, that children will have greater difficulty with quantitative comparisons than with qualitative ones.

An alternative hypothesis can be presented. It could be argued that children's difficulty with quantitative comparisons is rooted in their being unable to consider two variables at a time, that is, in being unable to deal with coordinated changes. In fact one can develop a strategy (see figure 1) to work correctly all of the qualitative comparisons without using the notion of proportion.

FIGURE 1

Strategy for comparison of ratios-One variable at a time


Which is lighter? a tablespoons of white sugar mixed with b tablespoons of brown sugar or a' tablespoons of white sugar mixed with b' tablespoons of brown sugar?
Look at the brown sugar
NO
Does one of the ratios have more white sugar?
YES
That ratio is lighter
The ratio with less brown sugar is the lighter

Children following this strategy will solve correctly all of the ratio comparisons of Table I, but will solve incorrectly most of the comparisons of Table II. We believe that the primary difficulty that many of these children have may be more fundamental than their apparent inability to make quantitative comparisons among ratios. We think that children have difficulty with coordinated changes, although there are certainly children whose difficulty with quantitative comparison of ratios does not lie in the coordination of change but in making a quantitative analysis.

In the research we report on here, we tried to find experiental evidence to support this hypothesis; that is, that there is a group of children who when making a comparison between two ratios coordinate the change among the two variables but do so in qualitative terms. Their difficulty lies in not being able to apply quantitative analysis to proportional reasoning.

Experiment 1

Subjects

Thirty six children between 10 and 13 years old (10 years N=9; 11 years N=11; 12 years N=11; 13 years N=5) participated in the study. Half of the children were girls and half were boys. The children were in 5th (N=12), 6th (N=12) and 7th (N=12) grades. Half of the children came from a school in a middle income neighborhood and half came from a school in a low income neighborhood.

Procedure

Each child worked individually two tasks in which he ordered pairs of ratios. Task 1 dealt with discrete quantities (patients per doctor in a hospital), and Task 2 dealt with continuous quantities (white sugar per brown sugar). Half of the children worked Task 1 first and half of the children worked Task 2 first.

Task 1 - Discrete quantities

The experimenter said to the child: "Suppose that these are doctors (cuisinaire 10 cm rods) and these are patients (cuisinaire 1 cm rods). There are W patients and B doctors in this hospital. In a second hospital here there are W' patients and B' doctors. In which of these hospitals does a doctor have more work? Why?"

Task 2 - Continuous quantities

The experimenter said to the child: "Here I have two containers. In this one I will mix W tablespoons of white sugar and B tablespoons of brown sugar. In the other container I will mix W' tablespoons of white sugar and B' tablespoons of brown sugar. If the sugar in each container is well mixed, which of the two containers will be lighter? Why?" (see Table III for the numbers used for B, B', W and W').

Results

Performance Level

There were no significant differences between the group of children who worked Task 1 first and those who worked it second. In addition, there was no difference between the performance of children from the school in the low-income neighborhood and those of the school in the middle-income neighborhood.

Qualitative vs. quantitative decision

Children had more difficulty comparing ratios in which a quantitative judgement was necessary than they had with those in which a qualitative decision was sufficient. When the comparison could be made on qualitative grounds 336 out of 360 times (93.3%) children answered it correctly. On the other hand, when the comparison involved coordinated change only 212 out of 432 times (49%) children solved it correctly.

Table III Performance Level

Comparison between		Discrete		Continuous	
		# Correct	%	# Correct	%
3/1	2/1	36	100%	35	97%
3/1	3/2	35	97%	31	86%
3/1	2/2	36	100%	32	88%

3/1	5/3	29	80%	8	22%
3/1	6/2	23	63.8%	2	5.5%
6/2	5/2	35	97%	32	94%
6/2	6/3	31	86%	31	86%
6/2	8/4	29	80%	10	27%
6/2	9/3	21	58%	1	2.7%
5/3	7/5	22	61%	16	44%
5/3	3/2	23	63%	28	77.7%

Some children solved correctly some of the problems requiring quantitative analysis by using (one variable at a time) qualitative strategy outlined earlier. There were two problems which were difficult to solve by qualitative methods since the ratios were equal (3/1, 6/2 and 6/2, 9/3). The fraction of such problems correctly solved was substantially smaller (47 correct out of 144 problems, 32.6%).

Discrete vs. continuous

Problems involving discrete quantities were significantly easier than those involving continuous quantities (Wilcoxon test $p \leq .001$). This difference was more pronounced on the comparison of ratios which involved quantitative decision (see Table III). In fact, among the children who solved correctly quantitative comparisons of problems involving continuous quantities only two children justified their decision on any kind of quantitative basis. The rest of the children used qualitative analysis. Consider for example the following protocol:

Subject (F, 11 year)

E : Which is lighter, 6 teaspoons of white sugar mixed with 2 teaspoons of brown sugar or 8 teaspoons of white sugar with 4 teaspoons of brown sugar?

S : 6 with 2

E : Why?

S : Although this one (8/4) has more white sugar it also has more brown sugar.

E : Which is lighter, 6 teaspoons of white sugar mixed with 2 teaspoons of brown sugar or 9 teaspoons of white sugar mixed with 3 teaspoons of brown sugar?

S : 9 with 3

E : Why?

S : Because there is more white sugar and the brown one is not so much more.

Type of ratio

The pairs of ratios that were compared can be divided into three categories:

1) One of the ratios has one in the denominator (e.g. 3/1, 5/3)

2) One of the ratios can be reduced to a ratio with one in the denominator (e.g. 6/2, 5/2)

3) Neither of the ratios can be reduced to a ratio with one in the denominator (e.g 5/3, 7/5)

There was no significant difference between children's performance in comparing pairs of ratios from category 1 and category 2.

On the other hand, there was a significant difference between the results of comparisons of ratios from category 3 and the comparisons of ratios from categories 1 and 2 (the comparisons in which the ratios were not equal). In the discrete case children's performance comparing (3/1, 5/3) and (6/2, 8/4) was significantly better than comparing either (5/3, 7/5) or (5/3, 3/2). In the continuous case the situation was reversed; children's performance comparing (5/3, 7/5) and (5/3, 3/2) was significantly better than their performance comparing (3/1, 5/3) or (6/2, 8/4).

Every time that children used quantitative proportional reasoning to justify their answer they reduced the ratios to ratios with one in the denominator. For example they reduced

(6/2, 8/4) to (3/1, 2/1).

This caused them some trouble when comparing 5/3 and 7/5 as well as 5/3 and 3/2. In this case some children made the following analysis (in the discrete case):

5 patients and 3 doctors means 2 patients are treated by the first doctor, 2 by the second doctor and 1 by the third doctor; while

7 patients and 5 doctors means 2 patients are treated by the first doctor, 2 by the second, and 1 each by the third, fourth and fifth doctors.

The children argued that because there was a greater number of doctors with only 1 patient in the second case, the doctors in the hospital with 5 patients and 3 doctors worked harder. Others children had difficulty reaching this conclusion. For example one child after he had made the previous analysis said:

"Here we have a problem, because some doctors attend 2 patients while others attend one. It is difficult to make a decision for a given hospital."

A second group of children who made quantitative arguments when comparing (3/1, 6/2) (6/2, 8/4) and (6/2, 9/3) switched to making qualitative arguments when dealing with (5/3, 7/3) and (5/3, 3/2).

Age

Although there were no significant differences in children's performance among the different age groups, (the number of children in each age group was small; 13 years N=5, 12 years N=11, 11 years N=11; 10 years N=9) we did find a trend towards increased use of quantitative judgements as children got older. Only 44% of the 10 years old group used quantitative analysis (at least once). Around 80% of the 11, 12 and 13 years old children used quantitatives analysis at least once.

Patterns of Performance

Almost all the children of this study solved correctly the tasks that called for qualitative comparison. In the quantitative comparison tasks four general patterns of performance were found:

(1) Children who always picked as lighter the ratio with more white sugar in it (or always the ratio with less brown sugar). These children seemed to focus on a single aspect of the situation at a time.

(2) Children who sometimes picked the ratio with less white sugar (e.g. pick 3/1 as lighter than 5/3) and other times picked the one with more white sugar (e.g. 6/2 as lighter than 3/1). These children probably have some understanding of the relationship between the elements that enter to form a ratio but when comparing ratios, based their reasoning on qualitative considerations.

(3) Children who compute the difference between the quantities of white sugar and compare this difference with the difference between the quantities of brown sugar. For example, they will say that 6/2 = 8/4 since 8-6 = 4-2. On the other hand they will say that 6/2 is lighter than 3/1, since 6-3 = 2-1.

(4) Children who solve the problems correctly. These children usually used a quantitative analysis of the situation.

Many children follow one pattern of performance on the task dealing with discrete quantities and another pattern on the task dealing with continuous quantities (see Table IV).

Table IV Patterns of Performance

	Discrete	Continuous
Single aspect of the situation at once	5	17
Qualitative analysis of the relationship	5	14

Additive analysis of the relationship	0	4
Quantitative proportional (at least once)	26	1
Total	36	36

The relationship between the patterns of the children's performance on the tasks and the types of justifications they offered for their answers will be discussed in the next section.

Type of Justifications

The students' explanation of their answers can be divided into several categories:

(1) Incomplete or illogical answers. Example: When comparing 3/1 and 5/3 they picked 5/3 and said: "...because there is more sugar".

(2) Explanations focussing on a single difference uncoordinated with other differences. Example: When comparing 3/1 and 5/3 they picked 5/3 and said: "it has more white sugar".

(3) Explanations focussing on both differences. These explanations can be divided into three subcategories:

(a) Explanations in which the analysis of the relationship between the two differences is made on a qualitative basis. Example: When comparing 3/1 and 5/3 they either pick 3/1 or 5/3, saying: "this one has more white sugar (5/3) but it also has more black sugar"

(b) Analysis where an additive principle was used. For example: When comparing 3/1 and 5/3 they said: "they are equal, because 5-3 = 3-1".

(c) Explanation using numerical quantification of the proportion. For example: When comparing 6/2 and 8/4 in the discrete case they said: "In the hospital with 6 patients and 2 doctors they work harder because each doctor has to attend 3 patients, while in the other hospital each doctor attends 2 patients"

(4) Finally there was a group of children who justified their answer by repeating the information given to them. For example: When comparing 3/1 and 5/3, after picking either of them they said:

"Because this one has 3 teaspoons of white sugar and 1 teaspoon of brown sugar and the other one has 5 teaspoons of white sugar and 3 teaspoons of brown sugar."

It is difficult to discern the nature of the reasoning that the children who gave these answers were using. In fact, we believe that

among the children giving this type of answer there are children using quite different types of reasoning.

Occasionally, the same child would use different types of justification in different problems. There is certain relationship, however, between the children's pattern of performance and the type of justification they used (see Table V). Leaving aside those instances in which the children repeated the problem as justification, one finds a systematic correspondence between the type of justification offered and the pattern of the child's performance. (see Table V). For example, children who made quantitative judgements at least once, justified their answers most of the time on the basis of a quantitative analysis.

Insert Table V about here

Discussion

The results of this study confirm prior findings that the notion of a ratio as a quantity is difficult for children. When deciding which of two ratios was bigger, it was significantly easier for children to pick the correct ratio when the decision could be reached on qualitative basis than when the decision needed quantitative analysis.

It has been already mentioned that the problems involving quantitative analysis also involve the coordination of two variables. It could be argued that the latter is the source of difficulty for the children and not the application of quantitative analysis.

There is a group of children (13% in Task 1; 47% in Task 2) whose main difficulty with the tasks that involve quantitative analysis might have been rooted in their lack of coordination of two variables. These children consistently picked the ratio with more white (less black) sugar on it.

Yet there is a group of children (13% in Task 1, 39% in Task 2) that also had difficulty with the problems involving quantitative analysis but who were probably coordinating change in a qualitative way. At least once, these children changed, in two consecutive problems, their criteria for picking the bigger ratio.

Sometimes their justifications clearly expressed the method of taking both elements into account but deciding on qualitative basis (see protocol above).

We can conclude then, that children originally deal with proportions in a qualitative way. The use of quantitative analysis in the comparison of ratio is a step in the development of proportional reasoning.

This step doesn't occur in a holistic way. Children apply quantitative reasoning more often to the tasks involving discrete quantities than to the ones involving continuous quantity; and to the ratios that could be reduced to ratios with "one" in the denominator (e.g. 6/3, 9/3, 8/4) than to those that couldn't be reduced (e.g. 5/3, 3/2).

These variations between tasks can shed light into the nature of children's

Table V

Relationship between patterns of performance & type of justification in Problems calling for quantitative judgements

A. Task I - Discrete Quantities

	Type of justification (number of times)									
	incomplete/ illogical		focus on a single variable		both variables qualitative		both variables additive		quantitative analysis		repeat the problem	
Focus on a single variable	9	30%	8	26.6%	1	3.3%	0		0		12	40%
Both variables qualitative	0		8	26.6%	12	40%	0		0		10	33.3%
Both variables additive	0		0		0		0		0		0	
Quantitative analysis	1	.6%	5	3.2%	23	14.7%	0		105	67.3%	22	14%

Table V

Relationship between patterns of performance & type of justification in

Problems calling for quantitative judgements

A. Task II - Continuous Quantities

	Type of justification (number of times)									
	incomplete/ illogical		focus on a single variable		both variables qualitative		both variables additive		quantitative analysis		repeat the problem	
Focus on a single variable	0		56	56%	12	12%	0		0		32	32%
Both variables qualitative	9	10.7%	40	47.6%	24	28.5%	0		0		11	13%
Both variables additive	0		1	5.5%	4	22.2%	11	61.1%			2	11.1%
Quantitative analysis	0		2	33.3%	0		0		4	66.6%	0	

ability to use quantitative analysis when dealing with ratio. This understanding can in turn help us develop teaching strategies to help children use proportional reasoning in various tasks.

For example, the fact that children used quantitative analysis more often when dealing with ratios like 6/2, 9/3 and 8/4 than when dealing with ratios like 5/3, 3/2 show that some knowledge of fractions is needed in order to work correctly the tasks. Now, how do children understand fractions? It is interesting to note that when comparing ratios using quantitative analysis children always reduced the fractions to fractions with one in the "denominator". Were they trying to avoid working with fractions? No child used a quantitative analysis which involve working directly with fractions with "denominator" different from one. For example, when comparing "6 patients for 2 doctors" with "8 patients for 4 doctors", no child ever said "6 patients for 2 doctors" is equivalent to "12 patients for 4 doctors" and therefore doctors in this hospital work more than in the hospital in which there are 8 patients for 4 doctors."

How is this fact related to their knowledge about and understanding of fractions? This area lends itself to further research.

A second result which suggests an area for further study is the fact that almost no one used proportional quantitative analysis in the task involving continuous quantity. What is the source of difficulty in this case? A possible explanation is that in the task involving discrete elements, the relationship between the elements was visible to the children while in the case of task with continuous elements the relationship was not visible since the two types of sugar were combined into a new type of sugar (light brown). If this is the case how can we help the children "see" the relationship in the continuous case? Can we use their understanding of proportions in the discrete case to help them understand the proportion in the continuous case?

The answers to these questions not only will increase our understanding of the development of the ratio concept in children but will also suggest possible activities for helping children learn these concepts.

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